A Jacobi-set based loss function for segmentation task

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February 7, 2023

Abstract

Segmentation of fine-scale structures in natural and bio-medical images are gaining importance with the development of high resolution electron microscopy images. The task still remains challenging as per-pixel accuracy is not only the metric of concern because of the imbalance in the dataset. In this project, a new loss function based on the Jacobi-sets are proposed.

1 Introduction

Motivation: Main motivation for this project comes from the paper titled "Topology-Aware Segmentation Using Discrete Morse Theory" by Hu et al. [1]. In the above mentioned paper, authors have defined a discrete Morse theory based loss function (*DMT-loss*) to identify correct topological structures and teaches a neural network to learn from these structures. They computed 1D-skeleton and 2D-patches (denoted by $S_1(\epsilon)$ and $S_2(\epsilon)$ respectively) from likelihood probability outputted by a neural network with discrete Morse theory[2] and compared with the ground truth to update the neural network parameters.

Related Work: In recent times persistent-homology-based losses were proposed by Hu et al.; Clough et al. [3, 4]. These methods identify a set of critical points of the likelihood function and the neural network (NN) to memorizes it. However it is inefficient for the NN to remember such a sparse set of critical points per iteration. Mosinska et al.in [5] uses pretrained filters to detect broken segments but their results are focused on 1D and cannot be generalized to higher dimensional structures. Deep neural networks are prevalent in the area of fine-scale structure segmentation and some notable works are [6, 7, 8, 5, 9]. Although discrete Morse theory has been used extensively to identify skeleton structures from images [10, 11, 12] , usage of Jacobi-sets [13] along with neural networks remains unexplored till date. In [13], Edelsbrunner et al. gave an algorithm to compute Jacobi-sets for multiple Morse functions. In [14] Jacobi-sets were used to compute combinatorial ridge-valley graphs and in [15] Bremer et al. used them to extract and track topological features. Bhatia et al. gave an algorithm to simplify Jacobi-sets locally and consistently in [16]. In [17], Tierny et al. used Jacobi-sets to compute bivariate Reeb spaces.

Contribution: We propose a new Jacobi-set based loss function. Compared to [1], our loss function does not depend on a parameter and runs in linear time. Furthermore, Jacobi-sets are infamous for their numerical instability. We propose a strategy such that when used with an end-to-end learnable framework, this instability is taken care of automatically by the learning process.

2 Background and Definitions

2.1 Background

Definition 2.1 (Jacobi-set)**.** Consider a smooth d-Manifold ($d \geq 2$) $\mathbb{M}$ equipped with a Riemannian metric so that gradients are defined. Consider two Morse functions $f, g : \mathbb{M} \to \mathbb{R}$. We denote the level set of g, $\mathbb{M}_t = g^{-1}(t)$, a smooth $(d-1)$ manifold for some $t \in \mathbb{R}$. The restriction of f on this level set is a smooth Morse function $f_t : \mathbb{M}_t \to \mathbb{R}$. The *Jacobi-set* $\mathbb{J}(f, g)$ is the set of critical points of such level set restrictions.

$$\mathbb{J}(f, g) = cl\big\{x \in \mathbb{M} \mid \text{ x is a critical point of } f_t\big\} \tag{1}$$

Intuitively *Jacobi-sets* can be thought as locus of critical points of f_t as g varies.

Consider the gradients of f, g at a point $x \in \mathbb{M}$ and $t = g(x)$. Then $\nabla f_t = 0$ iff gradient vectors of f and g are parallel, i.e.

$$\nabla f + \lambda \nabla g = 0 \text{ or } \lambda \nabla f + \nabla g = 0$$

. With this *Jacobi-sets* can be characterized as

$$\mathbb{J}(f, g) = cl\big\{x \in \mathbb{M} \mid \text{x is a critical point of} f + \lambda g \text{ or } \lambda f + g\big\} \tag{2}$$

for some $\lambda \in \mathbb{R}$.

2.2 Jacobi-sets for piecewise linear (PL) case:

$\mathbb{J}$ is generally computed by tracing the critical points of 1-parameter family of functions $h_\lambda = f + \lambda g$. As λ is varied, the critical points will move. Instead of keeping track of the movements and the critical points, in the PL setting we construct $\mathbb{J}$ as union of edges along which critical points move. Given a simplicial complex K, and uv an edge of K, as we vary λ, critical points of h_λ will move from u to v (or vice versa). If a critical point of h_λ moves along the edge uv,the following two conditions must hold

1. $h_\lambda(u) = h_\lambda(v)$
2. Entire edge uv must be critical.

Let $\lambda = \lambda_{uv}$ be the value when the first condition holds. Then

$$\begin{aligned} h_\lambda(u) &= h_\lambda(v) \\ f(u) + \lambda_{uv} g(u) &= f(v) + \lambda_{uv} g(v) \\ \lambda_{uv} &= \frac{f(v) - f(u)}{g(u) - g(v)} \end{aligned} \tag{3}$$

In PL setting criticality of an edge uv can be determined by the link condition. The link condition states that an edge uv is critical iff $\sum_{k\geq 0} \tilde{\beta}_{k-1} > 0$ for the *lower link* of uv with $\tilde{\beta}_i$ is the i-th reduced *betti* number. As this paper discusses only 2D examples, we will focus on the case where K is triangulation of a 2-manifold. Then each edge is incident to exactly two triangles discarding the boundary edges. Suppose triangles uvx and uvy is incident on the edge uv. Then the link of edge uv is the vertices x and y. Summarizing the above discussion, if K is a triangulation of 2-manifold, for an edge uv if its link is the vertices x and y, then uv belongs to Jacobi-set $\mathbb{J}$ iff

$$h_{\lambda_{uv}}(x) \times h_{\lambda_{uv}}(y) > 0 \tag{4}$$

with $\lambda_{uv} = \frac{f(v)-f(u)}{g(u)-g(v)}$ and $h_\lambda = f + \lambda g$. Adapting a notation used by Tierny et al. in [17], we denote $\overrightarrow{f}_{uv} = \langle f(u), f(v) \rangle$. We define $d_{uv} : \text{link}(uv) \to \mathbb{R}$, a signed distance function as

$$d_{uv}(v') = \langle \overrightarrow{f}_{vv'} \cdot \overrightarrow{\eta}_{f_{uv}} \rangle \tag{5}$$

Where v' is the vertex in link of uv, $\overrightarrow{\eta}_{f_{uv}}$ denotes a vector orthonormal to $\overrightarrow{f}_{uv}$ and $< a \cdot b >$ denotes dot-product between a and b. Then Eq 4 translates to

$$d_{uv}(x) \times d_{uv}(y) > 0 \tag{6}$$

with x and y being the link of the edge uv.

3 Method

Fig 1 shows the overview of our approach. Our loss has two terms, the binary cross-entropy loss L_{bce} and the *Jacobi loss*, L_J. So the overall loss-function is defined as $L(\hat{y}, y) = L_{bce}(\hat{y}, y) + L_J(\hat{y}, y)$ where $\hat{y}$ is the likelihood and y is the ground truth.

3.1 Jacobi loss

We chose **U-net** as the CNN architecture to generate the likelihood $\hat{y}$. Notice that $\hat{y}$ is defined on a 2D image, i.e. a grid $G \subset \mathbb{R}^2$ with $\hat{y} : G \to \mathbb{R}^2$. To compute Jacobi sets, we compute Freudenthal triangulation of G. Notice that this triangulation needs to be computed only once since the grid is of same

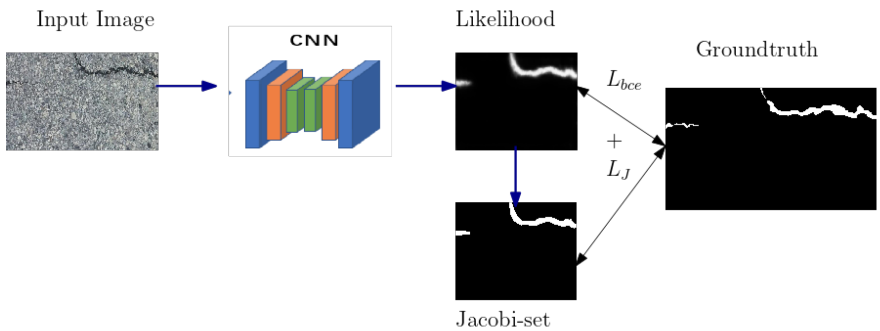


Figure 1: Overview of our loss function

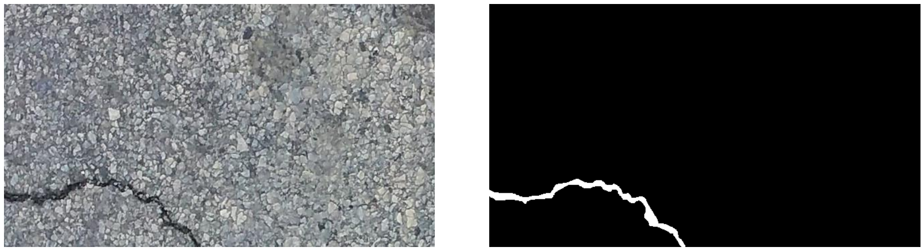
height and width across the training and test samples. Taking $\hat{y}$ to be one of the functions we compute Jacobi sets for the image. We take $\bigcup_{uv \in \mathbb{J}} uv = \{u\} \cup \{v\}$ we form the *Jacobi image* and compute Euclidean distance, l_2, from the ground truth. Notice that we still need $\overrightarrow{\eta}_{f_{uv}}$, a direction orthonormal to $\overrightarrow{f}_{uv}$. In other applications of Jacobi-sets this direction is generally pre determined. However, in our application we learn this direction.

3.2 Neural net architecture

The network architecture is similar to 2D U-net [6]. It consists of a contracting path and an expansive path (right side). The contracting path follows the typical architecture of a convolutional network. It consists of the repeated application of two 3x3 convolutions (unpadded convolutions), each followed by a rectified linear unit (ReLU) and a 2x2 max pooling operation with stride 2 for downsampling. At each downsampling step we double the number of feature channels. Every step in the expansive path consists of an upsampling of the feature map followed by a 2x2 convolution ("up-convolution") that halves the number of feature channels, a concatenation with the correspondingly cropped feature map from the contracting path, and two 3x3 convolutions, each followed by a ReLU. The cropping is necessary due to the loss of border pixels in every convolution. At the final layer a 1x1 convolution is used to map each 64- component feature vector to the desired number of classes. In total the network has 23 convolutional layers.

4 Experiments and Results

Dataset: We performed experiments on *CrackTree*[18] dataset. The dataset contains 1890 images. The images have dimension of 360×640 (height x width). Fig 2 shows a sample of training images.



(a) Sample input image.

(b) Groundtruth segmentation of (a).

Figure 2: Example images from the CrackTree dataset.

Evaluation metrics: Since per-pixel accuracy is not a valid metric we use DICE score (also known as DICE coefficient, DICE metric). DICE score is same as F1 score.

Training details: The U-net model was trained for 50 epochs without the *Jacobi loss* and with L_{bce} only. Then the same model was trained for another 50 epochs with $L_{bce} + L_J$. Another U-net model was trained for 100 epochs with just L_{bce} and was compared with the previous model.

Results: Fig 3 shows the result of segmented image. Table 1 shows the result

	Model with L_{bce}	Model with $L_{bce} + L_J$
DICE	0.657	0.616

Table 1: DICE score on CrackTree dataset with and without Jacobi loss.

of adding Jacobi loss on the CrackTree dataset.

5 Discussions and Conclusions

Due to time constraint extensive experiments could not be performed. We proposed a new loss function for this segmentation task. Although discrete Morse theory provides a robust, resilient to noise framework, Jacobi sets can be a good alternative since it can be computed efficiently. We also proposed a learnable framework for Jacobi set computation in this project.

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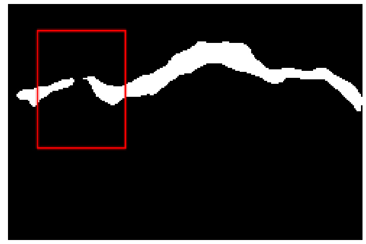
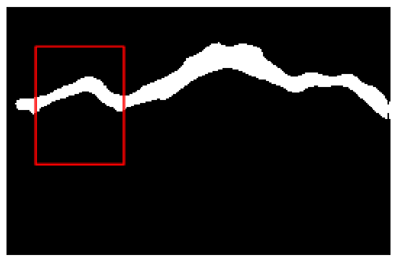
(a)



(b)



(c)

Figure 3: (a) Input image. (b) Segementation result with L_{bce} only. (c) Shows result with $L_{bce} + L_J$.

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